

第九章 拉普拉斯变换

§ 9.1 拉普拉斯变换的定义

一、拉氏正变换与拉氏反变换

1. 拉氏正变换

定义：

$$F(s) = \int_{0-}^{\infty} f(t) e^{-st} dt$$

式中 $s = \sigma + j\omega$ 为复数， $F(s)$ 称为 $f(t)$ 的象函数， $f(t)$ 称为 $F(s)$ 的原函数。

用符号 $\mathcal{L}[\]$ 表示对方括号里的时域函数作拉氏变换

即

$$F(s) = \mathcal{L}[f(t)]$$

2. 拉氏反变换

定义:
$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st} ds$$

用符号 $\mathcal{L}^{-1} []$ 表示对方括号里的复变函数作拉氏反变换

即
$$f(t) = \mathcal{L}^{-1} [F(s)]$$

拉氏变换对
$$f(t) \Leftrightarrow F(s)$$

二、应用拉氏变换求几种激励函数的象函数

1. 阶跃函数 $A\varepsilon(t)$

$$\mathcal{L} [A\varepsilon(t)] = \int_{0-}^{\infty} A\varepsilon(t) e^{-st} dt = \int_0^{\infty} A\varepsilon(t) e^{-st} dt = -\frac{A}{s} e^{-st} \Big|_0^{\infty} = \frac{A}{s}$$

$$\mathcal{L}^{-1} \left[\frac{A}{s} \right] = A\varepsilon(t)$$

$$A\varepsilon(t) \Leftrightarrow \frac{A}{s}$$

2. 冲激函数 $A\delta(t)$

$$\begin{aligned}\mathcal{L}[A\delta(t)] &= \int_{0-}^{\infty} A\delta(t)e^{-st} dt = \int_{0-}^{0+} A\delta(t)e^{-st} dt \\ &= \int_{0-}^{0+} A\delta(t)e^{-s \cdot 0} dt = A\end{aligned}$$

$$\mathcal{L}^{-1}[A] = A\delta(t)$$

$$A\delta(t) \Leftrightarrow A$$

3. 指数函数 $Ae^{-\alpha t}$

$$\begin{aligned}\mathcal{L}[Ae^{-\alpha t}] &= \int_{0-}^{\infty} Ae^{-\alpha t} \cdot e^{-st} dt = \int_{0-}^{\infty} Ae^{-(s+\alpha)t} dt \\ &= \frac{A}{-(s+\alpha)} e^{-(s+\alpha)t} \Big|_{0-}^{\infty} = \frac{A}{s+\alpha}\end{aligned}$$

$$\mathcal{L}^{-1}\left[\frac{A}{s+\alpha}\right] = Ae^{-\alpha t}$$

$$Ae^{-\alpha t} \Leftrightarrow \frac{A}{s+\alpha}$$

§ 9.2 拉普拉斯变换的基本性质

1. 线性性质

设 $f_1(t)$ 和 $f_2(t)$ 是两个任意的时间函数，它们的象函数分别为 $F_1(s)$ 和 $F_2(s)$ ， A_1 和 A_2 是两个任意常数，则

$$\begin{aligned}\mathcal{L}[A_1 f_1(t) + A_2 f_2(t)] &= A_1 \mathcal{L}[f_1(t)] + A_2 \mathcal{L}[f_2(t)] \\ &= A_1 F_1(s) + A_2 F_2(s)\end{aligned}$$

2. 微分性质

$$\text{若} \quad \mathcal{L}[f(t)] = F(s)$$

$$\text{则} \quad \mathcal{L}[f'(t)] = sF(s) - f(0_-)$$

3. 积分性质

若 $\mathcal{L}[f(t)] = F(s)$

则 $\mathcal{L}\left[\int_{0_-}^t f(\xi) \mathrm{d}\xi\right] = \frac{F(s)}{s}$

4. 延迟性质

若 $\mathcal{L}[f(t)] = F(s)$

则 $\mathcal{L}[f(t - t_0)] = \mathrm{e}^{-st_0} F(s)$

§ 9.3 拉普拉斯反变换的部分分式展开

若以 $F(s)$ 表示响应象函数，则

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_0 s^m + a_1 s^{m-1} + \cdots + a_m}{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}$$

一、 $F(s)$ 为真分式（ $m < n$ ）时的反变换

$D(s) = 0$ 的根可以是单根，共轭复根和重根三种情况

1. $D(s) = 0$ 有 n 个单根，分别为 $p_1, p_2, \cdots, p_n$

$$F(s) = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \cdots + \frac{K_n}{s - p_n}$$

$$(s - p_1)F(s) = K_1 + (s - p_1)\left(\frac{K_2}{s - p_2} + \cdots + \frac{K_n}{s - p_n}\right)$$

$$(s - p_1)F(s) = K_1 + (s - p_1)\left(\frac{K_2}{s - p_2} + \cdots + \frac{K_n}{s - p_n}\right)$$

$$K_1 = [(s - p_1)F(s)]_{s=p_1}$$

同理可求得 K_2 、 K_3 、... K_n 。

$$K_i = [(s - p_i)F(s)]_{s=p_i}$$

$\because p_i$ 是 $D(s) = 0$ 的一个根, $\therefore K_i$ 的表达式为 $\frac{0}{0}$ 的不定式

可以用求极限的方法确定 K_i 的值, 即

$$K_i = \lim_{s \rightarrow p_i} \frac{(s - p_i)N(s)}{D(s)} = \lim_{s \rightarrow p_i} \frac{(s - p_i)N'(s) + N(s)}{D'(s)} = \frac{N(p_i)}{D'(p_i)}$$

$$K_i = \frac{N(p_i)}{D'(p_i)}$$

例1

求 $F(s) = \frac{s+1}{2s^3 + 2s^2 - 4s}$ 的原函数

解：

方法一、
$$F(s) = \frac{s+1}{2s(s-1)(s+2)} = \frac{K_1}{s} + \frac{K_2}{s-1} + \frac{K_3}{s+2}$$

$$K_1 = sF(s)\Big|_{s=0} = s \cdot \frac{s+1}{2s(s-1)(s+2)}\Big|_{s=0} = -\frac{1}{4}$$

$$K_2 = (s-1)F(s)\Big|_{s=1} = \frac{s+1}{2s(s+2)}\Big|_{s=1} = \frac{1}{3}$$

同理
$$K_3 = -\frac{1}{12}$$

所以
$$f(t) = -\frac{1}{4} + \frac{1}{3}e^t - \frac{1}{12}e^{-2t} \quad t \geq 0$$

方法二、

$$F(s) = \frac{s+1}{2s^3 + 2s^2 - 4s} = \frac{K_1}{s} + \frac{K_2}{s-1} + \frac{K_3}{s+2}$$

$$D(s) = 2s^3 + 2s^2 - 4s$$

$$D'(s) = 6s^2 + 4s - 4$$

$$K_1 = \left. \frac{N(s)}{D'(s)} \right|_{s=0} = \left. \frac{s+1}{6s^2 + 4s - 4} \right|_{s=0} = -\frac{1}{4}$$

$$K_2 = \left. \frac{s+1}{6s^2 + 4s - 4} \right|_{s=1} = \frac{1}{3}$$

$$K_3 = \left. \frac{s+1}{6s^2 + 4s - 4} \right|_{s=-2} = -\frac{1}{12}$$

$$f(t) = -\frac{1}{4} + \frac{1}{3}e^t - \frac{1}{12}e^{-2t} \quad t \geq 0$$

2. $D(s) = 0$ 有共轭复根, 可采用配方的方法求反变换

例2

求 $F(s) = \frac{s+3}{s^2+2s+5}$ 的原函数

解:

$$\begin{aligned} F(s) &= \frac{s+3}{s^2+2s+1+4} = \frac{s+1+2}{(s+1)^2+2^2} \\ &= \frac{s+1}{(s+1)^2+2^2} + \frac{2}{(s+1)^2+2^2} \end{aligned}$$

所以

$$f(t) = e^{-t} \cos 2t + e^{-t} \sin 2t$$

思考:

求 $F(s) = \frac{s+4}{s^2+2s+5}$ 的原函数

3. $D(s) = 0$ 有重根

设 p_1 为 $D(s) = 0$ 的三重根, 其余为单根

$$F(s) = \frac{K_{13}}{s - p_1} + \frac{K_{12}}{(s - p_1)^2} + \frac{K_{11}}{(s - p_1)^3} + \left(\frac{K_2}{s - p_2} + \cdots \right)$$

$$(s - p_1)^3 F(s) = (s - p_1)^2 K_{13} + (s - p_1) K_{12} + K_{11} + (s - p_1)^3 \left(\frac{K_2}{s - p_2} + \cdots \right)$$

则
$$K_{11} = (s - p_1)^3 F(s) \Big|_{s=p_1}$$

$$\frac{d}{ds} [(s - p_1)^3 F(s)] = 2(s - p_1) K_{13} + K_{12} + \frac{d}{ds} [(s - p_1)^3 \left(\frac{K_2}{s - p_2} + \cdots \right)]$$

$$K_{12} = \frac{d}{ds} [(s - p_1)^3 F(s)] \Big|_{s=p_1}$$

同理可求得 $K_{13} = \frac{1}{2} \frac{d^2}{ds^2} [(s - p_1)^3 F(s)] \Big|_{s=p_1}$

推广至q重根

$$K_{11} = (s - p_1)^q F(s) \Big|_{s=p_1}$$

$$K_{12} = \frac{d}{ds} [(s - p_1)^q F(s)] \Big|_{s=p_1}$$

$$K_{13} = \frac{1}{2} \frac{d^2}{ds^2} [(s - p_1)^q F(s)] \Big|_{s=p_1}$$

• • • • •

$$K_{1q} = \frac{1}{(q-1)!} \frac{d^{q-1}}{ds^{q-1}} [(s - p_1)^q F(s)] \Big|_{s=p_1}$$

二、F(s)为假分式

$$F(s) = F_0(s) + \frac{N_0(s)}{D(s)}$$

例3

求 $F(s) = \frac{1}{(s+1)(s+2)^2}$ 的原函数

解:

$$F(s) = \frac{K_1}{s+1} + \frac{K_{22}}{s+2} + \frac{K_{21}}{(s+2)^2}$$

$$K_1 = (s+1)F(s)\Big|_{s=-1} = \frac{1}{(s+2)^2}\Big|_{s=-1} = 1$$

$$K_{21} = (s+2)^2 F(s)\Big|_{s=-2} = \frac{1}{s+1}\Big|_{s=-2} = -1$$

$$K_{22} = \frac{d}{ds} \frac{1}{s+1}\Big|_{s=-2} = -\frac{1}{(s+1)^2}\Big|_{s=-2} = -1$$

所以

$$f(t) = e^{-t} - e^{-2t} - te^{-2t}$$

§ 9.4 运算电路

一、基尔霍夫定律

时域形式

运算形式

对任意结点

$$\sum i(t) = 0$$

$$\sum I(s) = 0$$

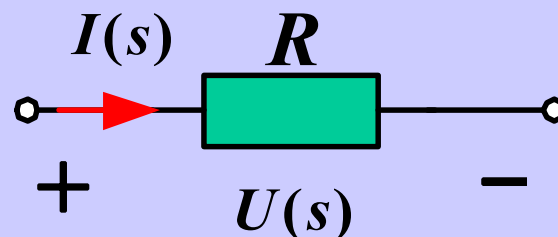
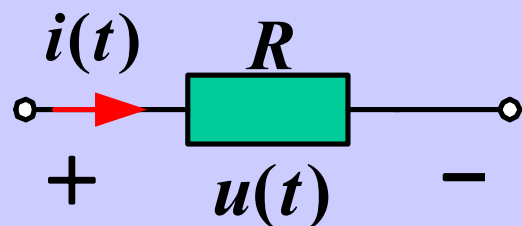
对任意回路

$$\sum u(t) = 0$$

线性性质

$$\sum U(s) = 0$$

二、电阻元件

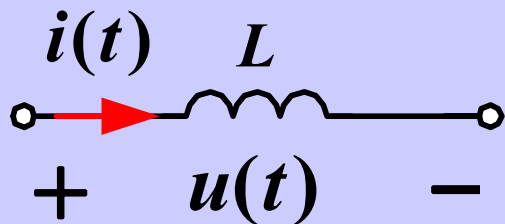


$$u(t) = Ri(t)$$

两边取拉斯变换，得

$$U(s) = RI(s)$$

三、电感元件



$$u(t) = L \frac{di(t)}{dt}$$

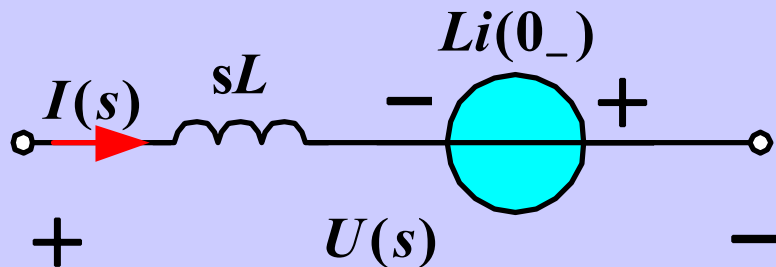
$$\mathcal{L}[u(t)] = \mathcal{L}\left[L \frac{di(t)}{dt}\right]$$

即

$$U(s) = sLI(s) - Li(0_-)$$

sL — 电感的运算阻抗

$i(0_-)$ — 电感的初始电流

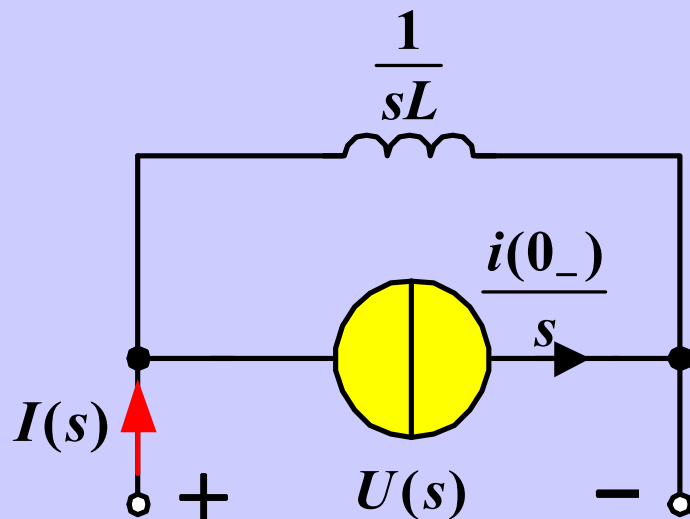


$Li(0_-)$ — 附加电压源的电压，
它反映了电感中初始电流的作用

电感的运算电路

$$U(s) = sLI(s) - Li(0_-) \Rightarrow$$

$$I(s) = \frac{1}{sL} U(s) + \frac{i(0_-)}{s}$$



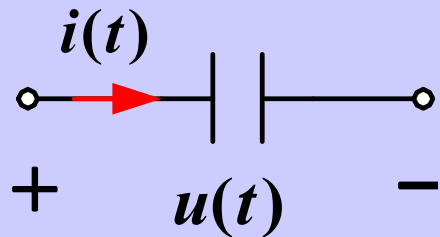
电感的运算电路

$\frac{1}{sL}$ — 电感的运算导纳

$\frac{i(0_-)}{s}$ — 附加电流源的电流

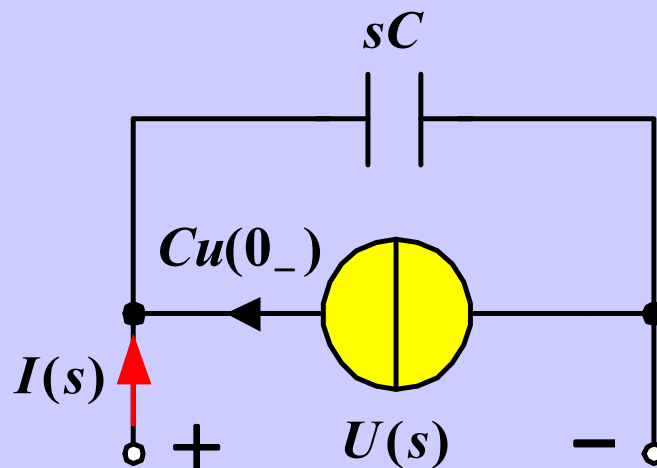
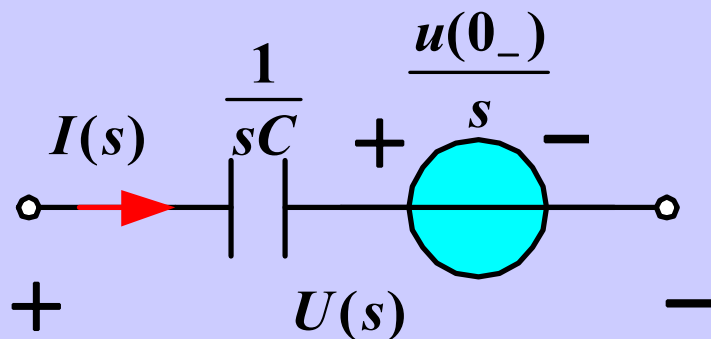
四、电容元件

$$u(t) = \frac{1}{C} \int_{0_-}^t i(t) dt + u(0_-)$$



$$U(s) = \frac{1}{sC} I(s) + \frac{u(0_-)}{s}$$

$$I(s) = sCU(s) - Cu(0_-)$$



电容的运算电路

$\frac{1}{sC}$ — 电容的运算阻抗

sC — 电容的运算导纳

$\frac{u(0_-)}{s}$ — 附加电压源的电压

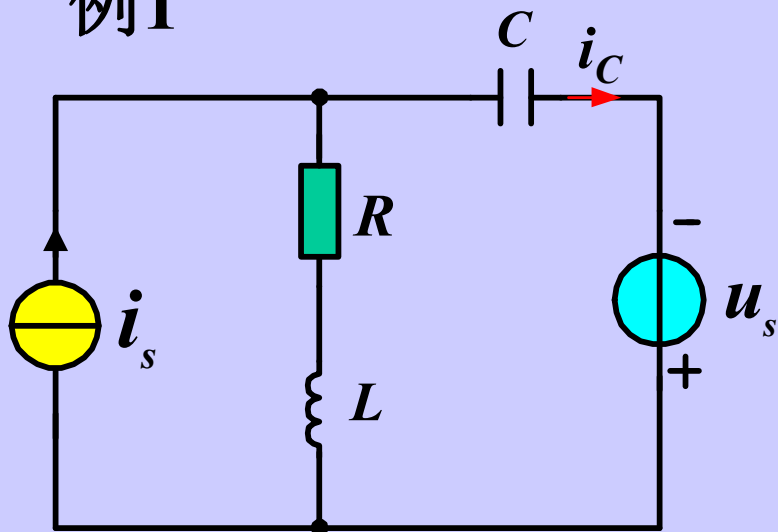
$Cu(0_-)$ — 附加电流源的电流

§ 13.4 应用拉普拉斯变换法 分析线性电路

一般步骤

1. 在 0_- 等效电路中, 确定 $u_C(0_-)$ 、 $i_L(0_-)$
2. 在换路后的电路中, 确定激励象函数 $F(s)$
3. 画出换路后的运算电路
4. 在运算电路中确定响应象函数
5. 进行拉氏反变换, 确定时域响应

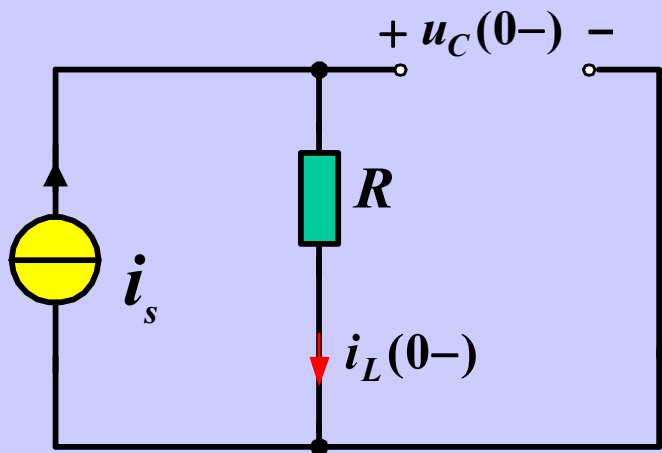
例1



电路如图，已知 $R = 3\Omega$, $L = 1\text{H}$,
 $C = 0.5\text{F}$, $i_s = 1\text{A}$, $u_s = e^{-3t}\varepsilon(t)\text{V}$.
 求电流 i_C

解：

1). 在 0_- 等效电路中，确定 $u_C(0_-)$ 、 $i_L(0_-)$



0_- 等效电路

$$i_L(0_-) = i_s = 1\text{A},$$

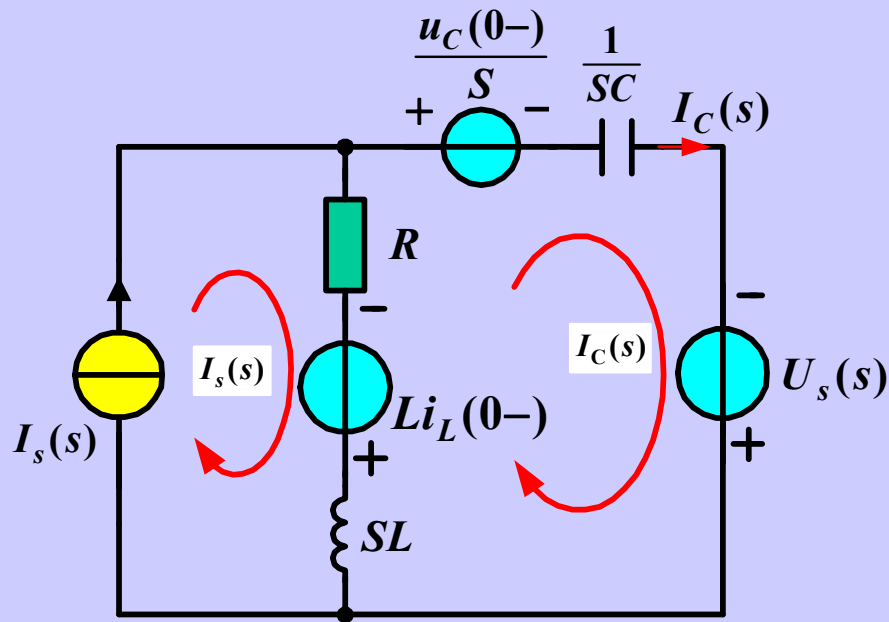
$$u_C(0_-) = Ri_s = 3 \times 1 = 3\text{V}$$

2). 确定激励象函数 $I_s(s)$ 、 $U_s(s)$

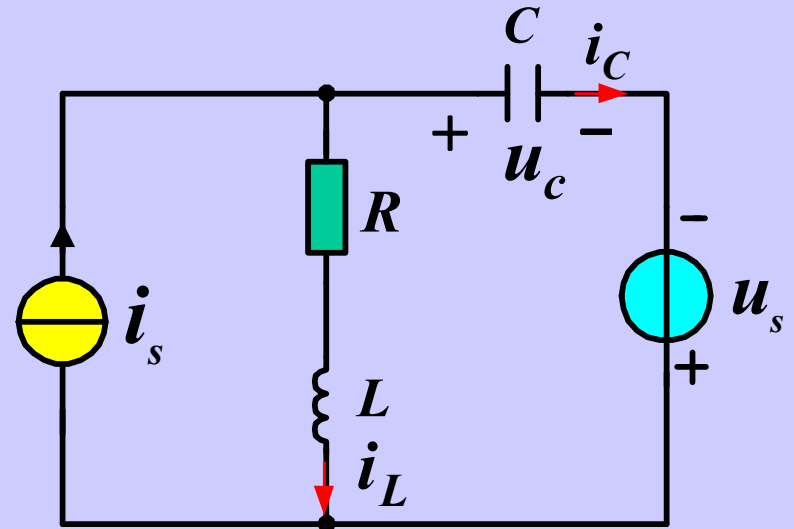
$$I_s(s) = \frac{1}{S}$$

$$U_s(s) = \frac{1}{S+3}$$

3) . 画出换路后的运算电路



运算电路



4) . 在运算电路中确定响应象函数

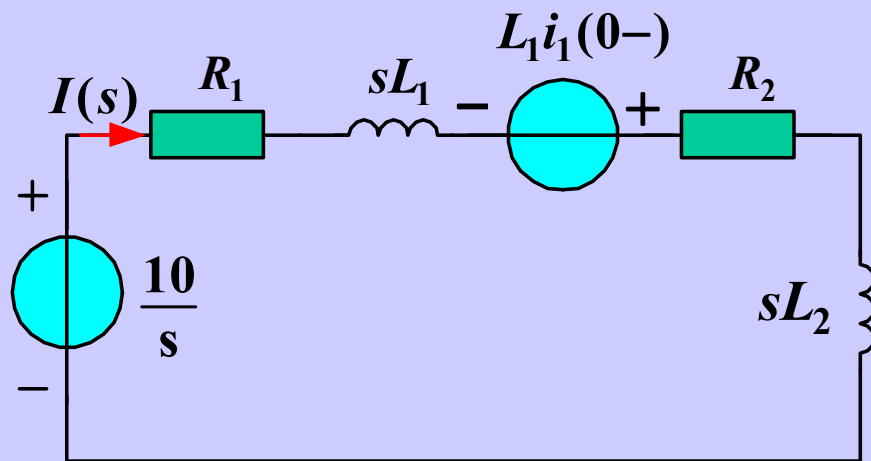
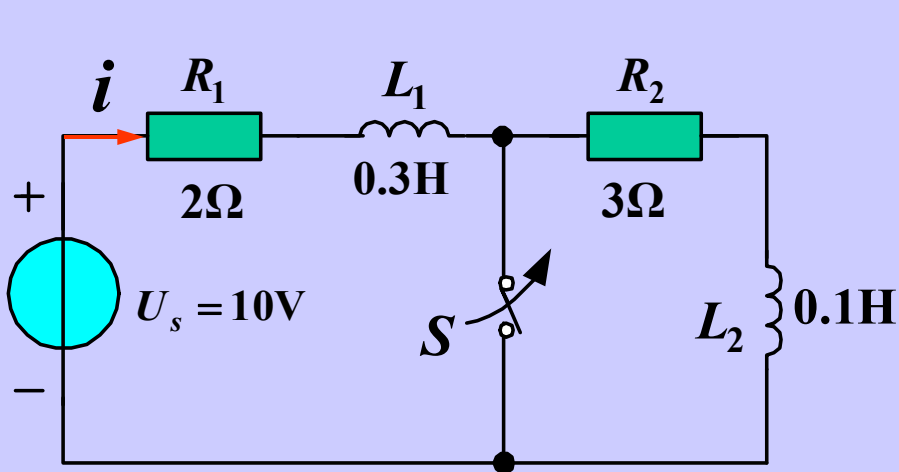
$$(R + \frac{1}{SC} + SL)I_C(s) - (R + SL)I_s(s) = U_s(s) - Li_L(0_-) - \frac{u_c(0_-)}{S}$$

$$I_c(s) = \frac{S}{(S+1)(S+2)(S+3)} = \frac{-0.5}{S+1} + \frac{2}{S+2} + \frac{-1.5}{S+3}$$

5). 确定时域响应 i_c

$$\begin{aligned} i_c = \mathcal{L}^{-1}[I_c(s)] &= \mathcal{L}^{-1}\left[\frac{-0.5}{S+1} + \frac{2}{S+2} + \frac{-1.5}{S+3}\right] \\ &= -0.5e^{-t} + 2e^{-2t} - 1.5e^{-3t} \quad A \quad t \geq 0 \end{aligned}$$

例2 电路如图所示，开关S原来闭合，求打开S后电路中的电流及电感元件的电压。



解：方法一、

运算电路

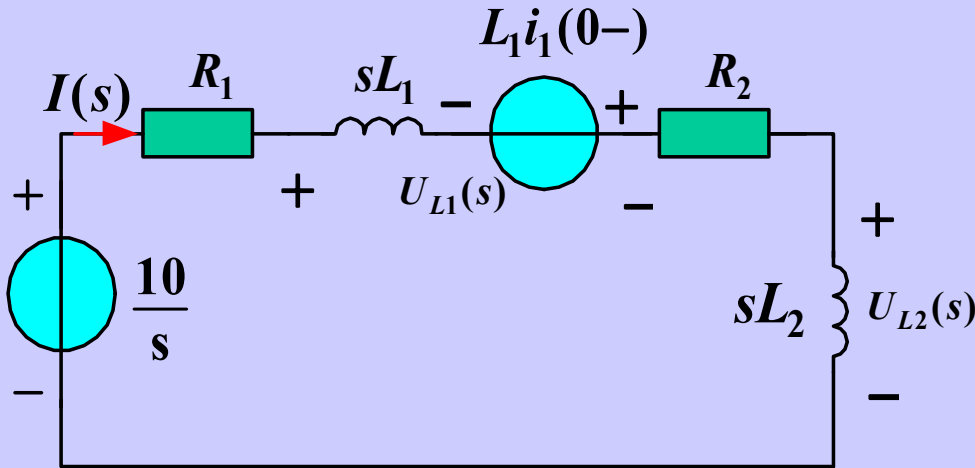
L_1 中的初始电流为 $\frac{U_s}{R_1} = 5\text{A}$, S打开后的运算电

路如图所示，故：

$$I(s) = \frac{\frac{10}{s} + 5L_1}{R_1 + R_2 + s(L_1 + L_2)} = \frac{10}{s(5 + 0.4s)} + \frac{1.5}{5 + 0.4s}$$

$$I(s) = \frac{25}{s(s+12.5)} + \frac{3.75}{s+12.5} = \frac{2}{s} + \frac{1.75}{s+12.5}$$

$$i(t) = (2 + 1.75e^{-12.5t})\text{A}$$



$$U_{L_1}(s) = 0.3sI(s) - 1.5$$

$$= 0.6 + \frac{0.3s \times 1.75}{s+12.5} - 1.5$$

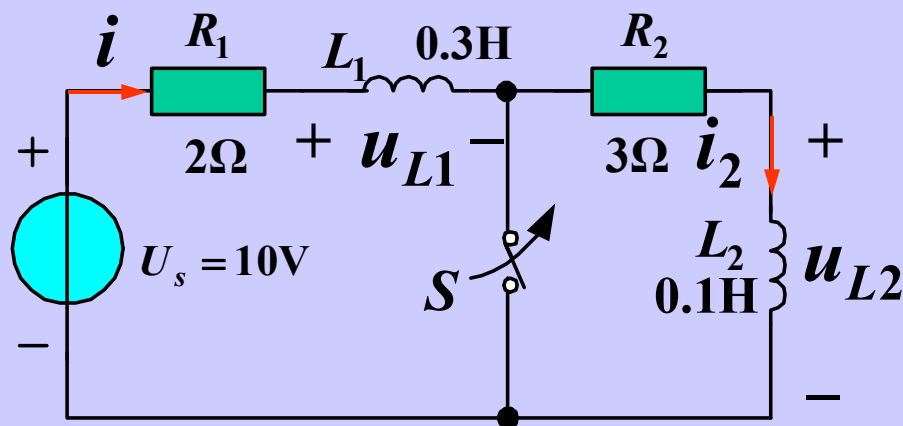
$$= -\frac{6.56}{s+12.5} - 0.375$$

$$U_{L_2}(s) = 0.1sI(s) = 0.2 + \frac{0.175s}{s+12.5} = -\frac{2.19}{s+12.5} + 0.375$$

$$u_{L_1}(t) = [-6.56e^{-12.5t} - 0.375\delta(t)]\text{V}$$

$$u_{L_2}(t) = [-2.19e^{-12.5t} + 0.375\delta(t)]\text{V}$$

方法二、



$$i_{L1}(0_-) = \frac{U_s}{R_1} = 5A$$

根据磁链守恒 $\psi(0_-) = \psi(0_+)$

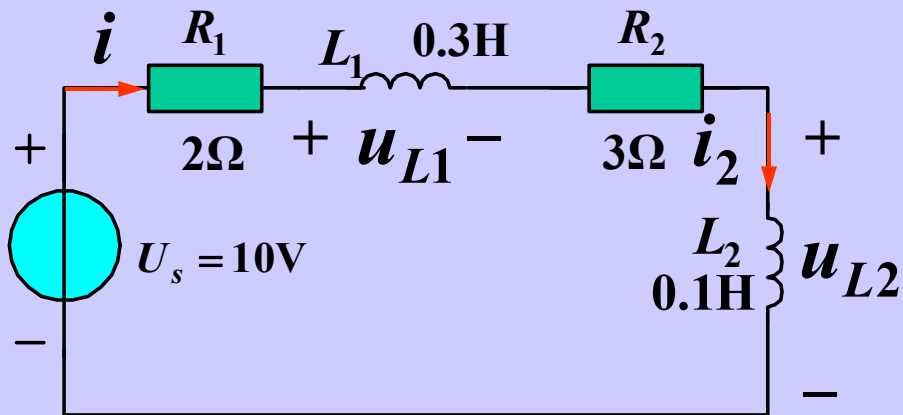
$$\begin{aligned}\psi(0_-) &= L_1 i_{L1}(0_-) \\ &= 0.3 \times 5 = 1.5\end{aligned}$$

$$\psi(0_+) = (L_1 + L_2) i(0_+) = 0.4 i(0_+) \quad \text{可得} \quad i(0_+) = 3.75A$$

$$\tau = \frac{L}{R} = \frac{L_1 + L_2}{R_1 + R_2} = \frac{0.4}{5} (s) \quad i(\infty) = \frac{10}{5} = 2A$$

$$\therefore i = 2 + (3.75 - 2)e^{-\frac{t}{\tau}} = 2 + 1.75e^{-12.5t} A \quad t > 0$$

$$i_2 = (2 + 1.75e^{-12.5t})\varepsilon(t) A$$



$$u_{L2} = L_2 \frac{di_{L2}}{dt} = -2.19e^{-12.5t} + 0.375\delta(t)V$$

$$u_{L1} = 10 - (R_1 + R_2)i - u_{L2} = -6.56e^{-12.5t} - 0.375\delta(t)V$$